

# Introduction to Khovanov Homologies.

## II. Reduced Jones superpolynomials

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### ABSTRACT

A second part of detailed elementary introduction into Khovanov homologies. This part is devoted to *reduced* Jones superpolynomials. The story is still about a hypercube of resolutions of a link diagram. Each resolution is a collection of non-intersecting cycles, and one associates a 2-dimensional vector space with each cycle. *Reduced* superpolynomial arises when for all cycles, containing a "marked" edge of the link diagram, the vector space is reduced to 1-dimensional. The rest remains the same. Edges of the hypercube are associated with cut-and-join operators, acting on the cycles. Superpartners of these operators can be combined into differentials of a complex, and superpolynomial is the Poincare polynomial of this complex. HOMFLY polynomials are practically the same in reduced and unreduced case, but superpolynomials are essentially different, already in the simplest examples of trefoil and figure-eight knot.

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This text is a continuation of [1], and is formulated not independently, but rather as a set of comments to that one. We assume that the reader is familiar with [1], use the same terminology and constructions, without going into lengthy explanations. All the references to relevant original works can be also found in [1]. References to formulas from [1] are given in the form (I.x) – in the present paper we refer to the version 1 (v1) of [1].

For relevant foundations of knot theory see [2]-[4]. For basic references on Khovanov-Rozansky homologies and superpolynomials see [5]-[21] and [22]-[41] respectively.

## 1 From unreduced to reduced Jones superpolynomial

### 1.1 Ordinary Jones

The unreduced Jones polynomial in the form (I.13) is obviously divisible by  $D = q + q^{-1}$ :

$$J^{\Gamma_c}(q) = (-)^{n_{\circ}} q^{n_{\bullet} - 2n_{\circ}} \sum_{\text{resolutions } r \text{ of } \Gamma} (-q)^{|r - r_c|} D^{\nu_r} = D \cdot \underline{J}^{\Gamma_c}(q) \quad (1)$$

The ratio  $\underline{J}^{\Gamma}(q)$  is called *reduced* Jones polynomial. In what follows we underline the variables (times and their superpartners), which are being reduced, as well as the objects (differential, cohomologies, knot polynomials) obtained *in result* of the reduction – hopefully, this does not cause confusion.

To obtain  $\underline{J}^{\Gamma}(q)$  from (I.13) one should say that one of the cycles at each vertex of the resolutions hypercube contributes not  $D$ , but just 1. For this purpose one can mark one *edge*  $E = e$  of  $\Gamma$  – in each resolution there will be exactly one cycle, containing  $e$ , – and let *this* cycle contribute 1 instead of  $D$ .

Since there is always an item in the sum (1) with  $\nu_r = 1$ , there is only one common power of  $D$  – and thus only one edge can be marked in  $\Gamma$  in above construction. Of course, the answer does not depend on the choice of this single  $e$ .

### 1.2 Reduced superpolynomial

Now it is clear, what should be done in Khovanov's  $T$ -deformation: one should reduce the vector spaces  $V$  in (I.33), associated with all the cycles, which contain the marked edge  $e$ , from two- to one-dimensional. This means that the corresponding  $\theta$ -variables in (I.46) should be nullified.

It is important here that the maps  $Q$  in (I.40) diminish the  $q$ -grading by one: because of this  $\theta$ -derivatives are always multiplied by  $\theta$  – what makes such reduction self-consistent. This would not be true if we tried to nullify  $\eta$ -variables instead.

In other words, nullification of underlined elements  $v_-$  in the following formulas is self-consistent, while it would not be like that if one attempts to nullify  $v_+$ :

$$\begin{array}{lll} \underline{V} \rightarrow \underline{V} \otimes V & & \\ v_+ \rightarrow v_+ \otimes v_- + \underline{v_-} \otimes v_+ & \implies & v_+ \rightarrow v_+ \otimes v_-, \\ \underline{v_-} \rightarrow \underline{v_-} \otimes v_- & \implies & 0 \rightarrow 0 \\ \underline{V} \otimes V \rightarrow \underline{V} & & \\ \underline{v_-} \otimes v_- \rightarrow 0 & \implies & 0 \rightarrow 0, \\ \underline{v_-} \otimes v_+ \rightarrow \underline{v_-} & \implies & 0 \rightarrow 0, \\ v_+ \otimes v_- \rightarrow \underline{v_-} & \implies & v_+ \otimes v_- \rightarrow 0, \\ v_+ \otimes v_+ \rightarrow v_+ & \implies & v_+ \otimes v_+ \rightarrow v_+ \end{array}$$

## 2 Example: Trefoil $3_1$ in a 2-strand realization

### 2.1 Hypercube, Jones and the cut-and-join operator

Extended Jones polynomial (I.7.4.1) is

$$\mathcal{J}^{\bullet\bullet\bullet} = \underline{p_3 p'_3} + t(\underline{p''_6} + \underline{p'_6} + \underline{p_6}) + t^2(\underline{p_4 p_2} + \underline{p'_4 p'_2} + \underline{p''_4 p''_2}) + t^3 \underline{p_2 p'_2 p''_2} \quad (2)$$

Underlined are the terms, eliminated by reduction – and there is exactly one such factor in each item.

This Jones polynomial and the cut and join operator below are build with the help of the hypercube quiver:



(which we do not underline after that). However, introduction of Grassmannian variables requires accurate work with the signs. We fix the basis vectors in the space of Grassmannian variables lexicographically:  $\theta'_2$  stands before  $\eta'_2$ , before  $\theta''_2$ , before  $\theta'_3$ ..., i.e. the basis vector is  $\eta'_2\theta_3$  rather than  $\theta_3\eta'_2$  or  $\theta_3\eta'_3$  rather than  $\eta'_3\theta_3$ . Note that the sign  $\ominus$  refers to the sign in the transformation matrix, the sign of the corresponding term in the differential depends on the way the variables are ordered in this term – and does not need to coincide with  $\ominus$  in bosonic form of the cut-and-join operators. The second derivative over Grassmann variables is defined in inverse order, as  $\frac{\partial^2}{\partial\theta\partial\eta} = \frac{\partial}{\partial\eta}\frac{\partial}{\partial\theta}$ , so that  $\frac{\partial^2}{\partial\theta\partial\eta}\theta\eta = +1$ .

For example, the differential

$$d_1 = (\eta_6 + \eta'_6 + \eta''_6) \frac{\partial^2}{\partial\eta_3\partial\eta'_3} + (\theta_6 + \theta'_6 + \theta''_6) \left( \frac{\partial^2}{\partial\eta_3\partial\theta'_3} + \frac{\partial^2}{\partial\theta_3\partial\eta'_3} \right) \quad (4)$$

acts on the basis vectors of the space  $\mathcal{C}_1$  as follows:

$$d_1 \downarrow \begin{array}{c|cccc} \mathcal{C}_1 & \theta_3\theta'_3 & \theta_3\eta'_3 & \eta_3\theta'_3 & \eta_3\eta'_3 \\ \hline d\mathcal{C}_1 & 0 & \theta_6 + \theta'_6 + \theta''_6 & \theta_6 + \theta'_6 + \theta''_6 & \eta_6 + \eta'_6 + \eta''_6 \end{array} \quad (5)$$

i.e. provides basis vectors of  $\mathcal{C}_2$  with all signs positive.

If, however, we take  $d_3$  the situation will be different. The naive superpartner to  $K_3$  would be

$$\theta_2\theta'_2 \frac{\partial}{\partial\theta''_4} \ominus \theta_2\theta''_2 \frac{\partial}{\partial\theta'_4} + \theta'_2\theta''_2 \frac{\partial}{\partial\theta_4} + (\theta_2\eta'_2 + \eta_2\theta'_2) \frac{\partial}{\partial\eta'_4} \ominus (\theta_2\eta''_2 + \eta_2\theta''_2) \frac{\partial}{\partial\eta_4} + (\theta'_2\eta''_2 + \eta'_2\theta''_2) \frac{\partial}{\partial\eta_4} \quad (6)$$

However such operator converts a basis vector  $\theta_2\theta_4$  into  $-\theta_2\theta'_2\theta''_2$ , which is minus the basis vector. At the same time another basis vector  $\theta'_2\theta'_4$  is converted into  $\oplus\theta_2\theta'_2\theta''_2$ , i.e. would be correct if we substitute  $-$  instead of  $\ominus$ . This means that the signs should be changed appropriately:

$$d_3 = - \left( \theta_2\theta'_2 \frac{\partial}{\partial\theta''_4} \hat{+} \theta_2\theta''_2 \frac{\partial}{\partial\theta'_4} + \theta'_2\theta''_2 \frac{\partial}{\partial\theta_4} + (\theta_2\eta'_2 + \eta_2\theta'_2) \frac{\partial}{\partial\eta'_4} \hat{+} (\theta_2\eta''_2 + \eta_2\theta''_2) \frac{\partial}{\partial\eta_4} + (\theta'_2\eta''_2 + \eta'_2\theta''_2) \frac{\partial}{\partial\eta_4} \right) \quad (7)$$

and it turns out that on the place of  $\ominus$  in this case one should substitute the same signs as everywhere else: we put hats over the two terms with  $\ominus$  which would naively have different signs. In eqs.(13) and (15) below we shall see that the hatted signs are absolutely necessary to guarantee the nilpotency property  $d_3d_2 = 0$ , in particular that  $\text{Im}(d_2) \subset \text{Ker}(d_3)$ .

After this comment we can return to our main line. The differential  $d_1$  is reduced as follows:

$$d_1 = (\eta''_6 + \eta'_6 + \eta_6) \frac{\partial^2}{\partial\eta_3\partial\eta'_3} + (\underline{\theta''_6} + \underline{\theta'_6} + \underline{\theta_6}) \left( \frac{\partial^2}{\partial\eta_3\partial\theta'_3} + \frac{\partial^2}{\partial\underline{\theta_3}\partial\eta'_3} \right) \implies \underline{d_1} = (\eta''_6 + \eta'_6 + \eta_6) \frac{\partial^2}{\partial\eta_3\partial\eta'_3} \quad (8)$$

Underlined here are the variables which should be nullified in the reduction. Obviously the only derivative w.r.t. an underlined variables ( $\underline{\theta_3}$ ) comes multiplied by the underlined variables.

The kernel and cohomology of  $d_1$  are changed by the reductions as follows:

$$\begin{aligned} \text{Ker}(d_1) &= \left\{ \underline{\theta_3}\theta_3, \eta_3\theta'_3 - \underline{\theta_3}\eta'_3 \right\} \implies \text{Ker}(\underline{d_1}) = \left\{ \eta_3\theta'_3 \right\}, \\ \dim_q(H_0) &= \dim_q \text{Ker}(d_1) = q^{-2} + 1 \implies \dim_q(\underline{H_0}) = \dim_q \text{Ker}(\underline{d_1}) = 1 \end{aligned} \quad (9)$$

and the image of  $d_1$  is

$$\text{Im}(d_1) = \left\{ \underline{\theta''_6} + \underline{\theta'_6} + \underline{\theta_6}, \eta''_6 + \eta'_6 + \eta_6 \right\} \implies \text{Im}(\underline{d_1}) = \left\{ \eta''_6 + \eta'_6 + \eta_6 \right\} \quad (10)$$

Similarly, for  $d_2$ :

$$\begin{aligned} d_2 &= \theta_2\underline{\theta_4} \left( \frac{\partial}{\partial\underline{\theta''_6}} - \frac{\partial}{\partial\underline{\theta'_6}} \right) + \underline{\theta'_2}\theta'_4 \left( \frac{\partial}{\partial\underline{\theta''_6}} - \frac{\partial}{\partial\underline{\theta'_6}} \right) + \theta''_2\underline{\theta''_4} \left( \frac{\partial}{\partial\underline{\theta''_6}} - \frac{\partial}{\partial\underline{\theta'_6}} \right) + \\ &+ (\underline{\eta_2}\theta_4 + \theta_2\eta_4) \left( \frac{\partial}{\partial\underline{\eta''_6}} - \frac{\partial}{\partial\underline{\eta'_6}} \right) + (\eta'_2\theta'_4 + \underline{\theta'_2}\eta'_4) \left( \frac{\partial}{\partial\underline{\eta''_6}} - \frac{\partial}{\partial\underline{\eta'_6}} \right) + (\eta''_2\underline{\theta''_4} + \theta''_2\underline{\eta''_4}) \left( \frac{\partial}{\partial\underline{\eta''_6}} - \frac{\partial}{\partial\underline{\eta'_6}} \right) \\ &\implies \underline{d_2} = \theta_2\eta_4 \left( \frac{\partial}{\partial\underline{\eta''_6}} - \frac{\partial}{\partial\underline{\eta'_6}} \right) + \eta'_2\theta'_4 \left( \frac{\partial}{\partial\underline{\eta''_6}} - \frac{\partial}{\partial\underline{\eta'_6}} \right) + \theta''_2\underline{\eta''_4} \left( \frac{\partial}{\partial\underline{\eta''_6}} - \frac{\partial}{\partial\underline{\eta'_6}} \right) \end{aligned} \quad (11)$$

so that

$$\begin{aligned} \text{Ker}(d_2) &= \left\{ \eta_6'' + \eta_6' + \eta_6, \underline{\theta_6''} + \underline{\theta_6'} + \underline{\theta_6} \right\} \implies \text{Ker}(\underline{d_2}) = \left\{ \eta_6'' + \eta_6' + \eta_6 \right\}, \\ H_1 &= \text{Ker}(d_2)/\text{Im}(d_1) = \emptyset \implies \underline{H_1} = \text{Ker}(\underline{d_2})/\text{Im}(\underline{d_1}) = \emptyset \end{aligned} \quad (12)$$

and

$$\begin{aligned} \text{Im}(d_2) &= \left\{ \theta_2 \underline{\theta_4} + \underline{\theta_2'} \theta_4', \theta_2'' \underline{\theta_4''} + \underline{\theta_2'''} \theta_4'', (\eta_2 \underline{\theta_4} + \theta_2 \eta_4) + (\eta_2' \theta_4' + \underline{\theta_2'} \eta_4'), (\eta_2'' \underline{\theta_4''} + \theta_2'' \eta_4'') + (\eta_2' \theta_4' + \underline{\theta_2'} \eta_4') \right\} \\ \implies \text{Im}(\underline{d_2}) &= \left\{ \theta_2 \eta_4 + \eta_2' \theta_4', \theta_2'' \eta_4'' + \eta_2' \theta_4' \right\} \end{aligned} \quad (13)$$

Finally, for  $d_3$ :

$$\begin{aligned} -d_3 &= \theta_2 \underline{\theta_2'} \frac{\partial}{\partial \theta_4''} \hat{+} \theta_2 \theta_2'' \frac{\partial}{\partial \theta_4'} + \underline{\theta_2'''} \theta_2'' \frac{\partial}{\partial \theta_4} + (\theta_2 \eta_2' + \eta_2 \underline{\theta_2'}) \frac{\partial}{\partial \eta_4''} \hat{+} (\theta_2 \eta_2'' + \eta_2 \theta_2'') \frac{\partial}{\partial \eta_4'} + (\underline{\theta_2'} \eta_2'' + \eta_2' \theta_2'') \frac{\partial}{\partial \eta_4} \\ \implies -\underline{d_3} &= \hat{+} \theta_2 \theta_2'' \frac{\partial}{\partial \theta_4'} + \theta_2 \eta_2' \frac{\partial}{\partial \eta_4''} \hat{+} (\theta_2 \eta_2'' + \eta_2 \theta_2'') \frac{\partial}{\partial \eta_4'} + \theta_2' \eta_2' \frac{\partial}{\partial \eta_4} \end{aligned} \quad (14)$$

so that

$$\begin{aligned} \text{Ker}(d_3) &= \left\{ \theta_2 \underline{\theta_4} \hat{+} \underline{\theta_2'} \theta_4', \theta_2'' \underline{\theta_4''} \hat{+} \underline{\theta_2'''} \theta_4'', \theta_2 \eta_4 \hat{+} \eta_2' \theta_4' - \eta_2'' \theta_4'', \theta_2'' \eta_4'' \hat{+} \eta_2' \theta_4' - \eta_2 \theta_4, \eta_2 \underline{\theta_4} \hat{+} \underline{\theta_2'} \eta_4' + \eta_2'' \theta_4'' \right\} \\ \implies \text{Ker}(\underline{d_3}) &= \left\{ \theta_2 \eta_4 \hat{+} \eta_2' \theta_4', \theta_2'' \eta_4'' \hat{+} \eta_2' \theta_4', \eta_2 \eta_4 \hat{+} \eta_2' \eta_4' + \eta_2'' \eta_4'' \right\}, \\ H_2 &= \text{Ker}(d_3)/\text{Im}(d_2) = \left\{ \eta_2 \underline{\theta_4} \hat{+} \underline{\theta_2'} \eta_4' + \eta_2'' \theta_4'' \right\} \\ \implies \underline{H_2} &= \text{Ker}(\underline{d_3})/\text{Im}(\underline{d_2}) = \left\{ \eta_2 \eta_4 + \eta_2' \eta_4' + \eta_2'' \eta_4'' \right\}, \\ \dim_q(H_2) &= q^0 = 1, \quad \dim_q(\underline{H_2}) = q^2, \end{aligned} \quad (15)$$

and

$$\begin{aligned} \text{Im}(d_3) &= \left\{ \theta_2 \underline{\theta_2'} \theta_2'', \eta_2 \underline{\theta_2'} \theta_2'', \theta_2 \eta_2' \theta_2'', \theta_2 \underline{\theta_2'} \eta_2'', \theta_2 \eta_2' \eta_2'', \eta_2 \eta_2' \theta_2'', \eta_2 \underline{\theta_2'} \eta_2'' \right\} \\ \implies \text{Im}(\underline{d_3}) &= \left\{ \theta_2 \eta_2' \theta_2'', \eta_2 \eta_2' \theta_2'', \theta_2 \eta_2' \eta_2'' \right\} \end{aligned} \quad (16)$$

i.e.

$$\underline{H_3} = \text{Coim}(\underline{d_3}) = \left\{ \eta_2 \eta_2' \eta_2'' \right\}, \quad \dim_q(\underline{H_3}) = q^3 \quad (17)$$

and this time this coincides with

$$H_3 = \text{Coim}(d_3) = \left\{ \eta_2 \eta_2' \eta_2'' \right\}, \quad \dim_q(H_3) = q^3 \quad (18)$$

In fact the boxed arrow above is not naive. Note that in variance with the cases of  $d_1$  and  $d_2$  the dimensions of kernels and images are not just divided by two when the reduction is performed: it is enough to say that for  $d_3$  these dimensions are odd. Here we encounter a more serious reshuffling. It is implied by the fact that

$$d_3 \left( \eta_2 \eta_4 + \eta_2' \eta_4' + \eta_2'' \eta_4'' \right) = 2 \eta_2 \underline{\theta_2'} \eta_2'' \quad (19)$$

In the image of  $d_3$  this fact is reflected just by nullification of  $\eta_2 \theta_2' \eta_2'' \in \text{Im}(d_3)$ , which thus drops away from  $\text{Im}(\underline{d_3})$ . However, in the kernels this looks slightly more involved: this combination of the  $q$ -grading level 2 did not belong to  $\text{Ker}(d_3)$ , but it appears  $\text{Ker}(\underline{d_3})$  and starts contributing to  $\underline{H_2}$ . Instead, the terms of  $q$ -grading 0, which contributed to  $H_2$ , are nullified and disappear from  $\underline{H_2}$ . This has a drastic impact on the form of the two superpolynomial, unreduced and reduced: the powers of  $q$  in front of  $T^2$  will be different in these two cases.

### 2.3 The Jones superpolynomial

In result unreduced Jones superpolynomial is

$$\begin{aligned}
 P^{3_1}(q|T) &= q^3 \sum_{j=0}^3 (qT)^j \dim_q(H_j) = \\
 &= q^3 \left( (q^{-2} + 1) \cdot (qT)^0 + 0 \cdot (qT)^1 + q^0 (qT)^2 + q^3 (qT)^3 \right) = q + q^3 + q^5 T^2 + q^9 T^3
 \end{aligned} \tag{20}$$

while the reduced one is

$$\begin{aligned}
 \underline{P}^{3_1}(q|T) &= q^{-1} \cdot q^3 \sum_{j=0}^3 (qT)^j \dim_q(\underline{H}_j) = \\
 &= q^2 \left( 1 \cdot (qT)^0 + 0 \cdot (qT)^1 + q^2 (qT)^2 + q^3 (qT)^3 \right) = \boxed{q^2 + q^6 T^2 + q^8 T^3}
 \end{aligned} \tag{21}$$

and

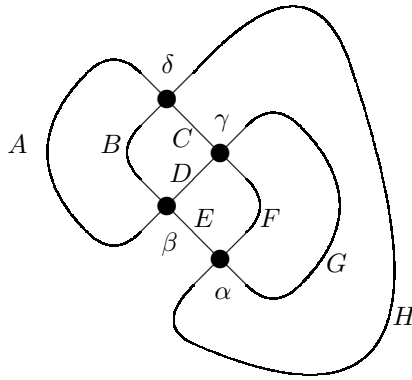
$$\boxed{P^{3_1} = (q + q^{-1}) \underline{P}^{3_1} - q^7 (1 + T) T^2} \tag{22}$$

The extra factor  $q^{-1}$  in the definition of  $\underline{P}^{3_1}(q|T)$  compensates for non-trivial  $q$ -grading power of the reduced 1-dimensional space (we keep  $v_+$  with the grading  $q$  as its basis element, while it should be rather shifted to 1). Minus sign in front of the last "correction term" shows that the *unreduced* superpolynomial is "more refined": the  $T$ -deformation of reduced Jones, after multiplication by  $D/q$ , can be further reduced – to provide a "smaller" superpolynomial, which is a deformation of the unreduced Jones.

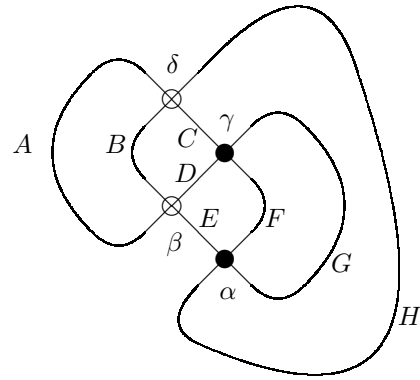
## 3 Reduced Jones superpolynomial for the figure-eight knot

### 3.1 Braid

The trefoil can be made not only from the 2-strand braid with 3 crossings, but also from the 3-strand braid with 4 crossings. Remarkably, in Khovanov formalism a similar representation for the non-torus figure-eight knot  $4_1$  differs only by the change of coloring.



$3_1$  knot (trefoil)



$4_1$  knot (figure eight)

The graph has eight edges and four vertices, labeled by capital latin and small green letters respectively. The hypercube is 4-dimensional and has 16 vertices: the graph possesses 16 resolutions. These involve  $2 + 4 + 1 + 4 + 4 + 5 = 20$  different cycles of the lengths 2, 3, 4, 5, 6, 8.

Drawing a four-dimensional hypecube is not very informative, therefore we substitute a picture by a table. The resolutions (hypecube vertices) are separated by double horizontal lines into sets with the same  $|r - r_c|$  from  $r_c = I$ , associated with the  $3_1$  knot. We also explicitly list the vertices, where flips are made to obtain the resolution from  $I$ .

resolution	$r$	flips at	cycles			
$I$	[0000]		$p_2 = AB$ $p'_2 = FG$	$p_4 = CDEH$		
$II$	[1000]	$\alpha$	$p_2 = AB$		$p_6 = CDEFGH$	
$III$	[0100]	$\beta$	$p'_2 = FG$		$p'_6 = ABCHE$	
$IV$	[0010]	$\gamma$	$p_2 = AB$		$p''_6 = CGFDEH$	
$V$	[0001]	$\delta$	$p'_2 = FG$		$p'''_6 = ABCDEH$	
$VI$	[1100]	$\alpha, \beta$				$p'_8 = ABDCHGFE$
$VII$	[1010]	$\alpha, \gamma$	$p_2 = AB$	$p_3 = DEF$ $p''_3 = CGH$		
$VIII$	[1001]	$\alpha, \delta$				$p_8 = ABCDEFGH$
$IX$	[0110]	$\beta, \gamma$				$p''_8 = ABDFGCHE$
$X$	[0101]	$\beta, \delta$	$p'_2 = FG$	$p'_3 = AEH$ $p'''_3 = BCD$		
$XI$	[0011]	$\gamma, \delta$				$p'''_8 = ABCGFDEH$
$XII$	[1110]	$\alpha, \beta, \gamma$		$p''_3 = CGH$	$p''_5 = ABDFE$	
$XIII$	[1101]	$\alpha, \beta, \delta$		$p'''_3 = BCD$	$p'''_5 = AHGFE$	
$XIV$	[1011]	$\alpha, \gamma, \delta$		$p_3 = DEF$	$p_5 = ABCGH$	
$XV$	[0111]	$\beta, \gamma, \delta$		$p'_3 = AEH$	$p'_5 = BCGFD$	
$XVI$	[1111]	$\alpha, \beta, \gamma, \delta$				$p''''_8 = AHGCBDFE$

For  $4_1$  the starting resolution is different:  $r_c = X$ , and the sets are different:

$ r - r_c $	resolutions $r$
0	$X$
1	$III, V, XIII, XV$
2	$I, VI, VIII, IX, XI, XVI$
3	$II, IV, XII, XIV$
4	$VII$

(23)

### 3.2 Jones polynomials

From these tables we immediately read the extended and unreduced Jones polynomials:

$$\begin{aligned} \mathcal{J}^{3_1} = & p_2 p'_2 p_4 + t \left( p_2 p_6 + p'_2 p'_6 + p_2 p''_6 + p'_2 p'''_6 \right) + \\ & + t^2 \left( p_2 p_3 p''_3 + p'_2 p'_3 p'''_3 + p_8 + p'_8 + p''_8 + p'''_8 \right) + t^3 \left( p_3 p_5 + p'_3 p'_5 + p''_3 p''_5 + p'''_3 p'''_5 \right) + t^4 p''''_8 \end{aligned} \quad (24)$$

We remind that *extended* knot polynomials are not topological invariants, in particular this  $\mathcal{J}^{3_1}$ , defined through the 3-strand braid, is drastically different from  $\mathcal{J}^{\bullet\bullet\bullet}$  in (2), defined for the 2-strand braid. However, the ordinary Jones polynomials, obtained after substitution (I.19) of  $p_k = D$  and  $t = -q$  are, of course, the same:

$$\begin{aligned} J^{\bullet\bullet\bullet}(q) = J^{3_1}(q) &= q^4 \left( D^3 - 4qD^2 + q^2(2D^3 + 4D) - 4q^3D^2 + q^4D \right) = \\ &= D \cdot q^4 \left( D^2 - 4qD + q^2((2D^2 + 4) - 4q^3D + q^4) \right) = q + q^3 + q^5 - q^9 = D \cdot (q^2 + q^6 - q^8) \end{aligned} \quad (25)$$

Similarly,

$$\begin{aligned} \mathcal{J}^{4_1} = & p'_2 p'_3 p'''_3 + t \left( p'_2(p'_6 + p'''_6) + p'_3 p'_5 + p'''_3 p'''_5 \right) + \\ & + t^2 \left( p_2 p'_2 p_4 + p_8 + p'_8 + p''_8 + p'''_8 + p''''_8 \right) + t^3 \left( p_2(p_6 + p''_6) + p_3 p_5 + p'_3 p'_5 \right) + t^4 p_2 p_3 p''_3 \end{aligned} \quad (26)$$

and this implies

$$\begin{aligned} J^{4_1}(q) &= \frac{q^2}{(-q^2)^2} \left( D^3 - 4qD^2 + q^2(D^3 + 5D) - 4q^3D^2 + q^4D^3 \right) = \\ &= D \cdot q^{-2} \left( D^2 - 4qD + q^2(D^2 + 5) - 4q^3D + q^4D^2 \right) = q^5 + q^{-5} = D(q^{-4} - q^{-2} + 1 - q^2 + q^4) \end{aligned} \quad (27)$$

### 3.3 Cut-and-join operators for the $3_1$ knot

To construct superpolynomials, unreduced and reduced, we need also the differentials. According to the general procedure, outlined in secs.6 and 7 of [1], from (26) we read the bosonic cut-and-join operators:

$$\begin{aligned} K_1 &= (p_6 + p''_6) \frac{\partial^2}{\partial p'_2 \partial p_4} + (p'_6 + p'''_6) \frac{\partial^2}{\partial p_2 \partial p_4} \\ K_2 &= \underbrace{p_3 p''_3 \left( \frac{\partial}{\partial p'_6} \ominus \frac{\partial}{\partial p_6} \right)}_{\alpha\gamma} + \underbrace{p'_3 p'''_3 \left( \frac{\partial}{\partial p''_6} \ominus \frac{\partial}{\partial p'_6} \right)}_{\beta\delta} + \underbrace{p_8 \left( \frac{\partial^2}{\partial p'_2 \partial p'''_6} \ominus \frac{\partial^2}{\partial p_2 \partial p_6} \right)}_{\alpha\delta} + \\ &+ \underbrace{p'_8 \left( \frac{\partial^2}{\partial p'_2 \partial p'_6} \ominus \frac{\partial^2}{\partial p_2 \partial p_6} \right)}_{\alpha\beta} + \underbrace{p''_8 \left( \frac{\partial^2}{\partial p_2 \partial p'_6} \ominus \frac{\partial^2}{\partial p'_2 \partial p'_6} \right)}_{\beta\gamma} + \underbrace{p'''_8 \left( \frac{\partial^2}{\partial p'_2 \partial p'''_6} \ominus \frac{\partial^2}{\partial p_2 \partial p'_6} \right)}_{\gamma\delta} \\ K_3 &= \underbrace{p''_3 p'_5 \left( \frac{\partial}{\partial p'_8} + \frac{\partial}{\partial p''_8} \right) \ominus p''_5 \frac{\partial^2}{\partial p_2 \partial p_3}}_{\alpha\beta\gamma} + \underbrace{p'''_3 p'''_5 \left( \frac{\partial}{\partial p'_8} \ominus \frac{\partial}{\partial p_8} \right) + p'_5 \frac{\partial^2}{\partial p'_2 \partial p'_3}}_{\alpha\beta\delta} + \\ &+ \underbrace{p_3 p_5 \left( \frac{\partial}{\partial p'''_8} \ominus \frac{\partial}{\partial p_8} \right) + p_5 \frac{\partial^2}{\partial p_2 \partial p'_3}}_{\alpha\gamma\delta} + \underbrace{p'_3 p'_5 \left( \frac{\partial}{\partial p'_8} + \frac{\partial}{\partial p'''_8} \right) \ominus p'_5 \frac{\partial^2}{\partial p'_2 \partial p'''_3}}_{\beta\gamma\delta} \end{aligned}$$



$$K_4 = p_8''' \left( \ominus \frac{\partial^2}{\partial p_3 \partial p_5} + \frac{\partial^2}{\partial p_3' \partial p_5'} \ominus \frac{\partial^2}{\partial p_3'' \partial p_5''} + \frac{\partial^2}{\partial p_3''' \partial p_5'''} \right) \quad (28)$$

Each item in the cut-and-join operator is associated with an edge of the hypercube, i.e. with a flip, made at exactly one vertex of original graph. By  $\ominus$  in this formula we denote the terms where the sign factor in the super-analogue of the cut-and-join operator will be negative,  $\epsilon_{bc}^a = -1$ , according to the rule (I.45). There are 12 (out of 32) edges of this type.

### 3.4 Cut-and-join operators for the $4_1$ knot

Similarly, for the  $4_1$  knot we have:

$$\begin{aligned} K_1 &= (p_6''' \ominus p_6') \frac{\partial^2}{\partial p_3' \partial p_3'''} \ominus p_5' \frac{\partial^2}{\partial p_2' \partial p_3'''} + p_5''' \frac{\partial}{\partial p_2' \partial p_3'} \\ K_2 &= \underbrace{p_2 p_4 \frac{\partial}{\partial p_6'} + p_2 p_4 \frac{\partial}{\partial p_6''}}_I + \underbrace{p_8 \left( \frac{\partial^2}{\partial p_2' \partial p_6'''} \ominus \frac{\partial^2}{\partial p_3''' \partial p_5'''} \right)}_{\alpha\delta} + \underbrace{p_8' \left( \frac{\partial^2}{\partial p_2' \partial p_6'} + \frac{\partial^2}{\partial p_3''' \partial p_5'''} \right)}_{\alpha\beta} + \\ &+ \underbrace{p_8'' \left( \frac{\partial^2}{\partial p_3' \partial p_5'} \right) \ominus \frac{\partial^2}{\partial p_2' \partial p_6'}}_{\beta\gamma} + \underbrace{p_8''' \left( \frac{\partial^2}{\partial p_2' \partial p_6'''} + \frac{\partial^2}{\partial p_3' \partial p_5'} \right)}_{\gamma\delta} + \underbrace{p_8'''' \left( \frac{\partial^2}{\partial p_3' \partial p_5'} + \frac{\partial^2}{\partial p_3''' \partial p_5'''} \right)}_{\alpha\beta\gamma\delta} \\ K_3 &= \underbrace{p_6 \frac{\partial^2}{\partial p_2' \partial p_4} \ominus p_2 p_6 \left( \frac{\partial}{\partial p_8} + \frac{\partial}{\partial p_8'} \right)}_{\alpha} + \underbrace{p_6'' \frac{\partial^2}{\partial p_2' \partial p_4} + p_2 p_6'' \left( \frac{\partial}{\partial p_8'} \ominus \frac{\partial}{\partial p_8''} \right)}_{\gamma} + \\ &+ \underbrace{p_3'' p_5'' \left( \frac{\partial}{\partial p_8'} + \frac{\partial}{\partial p_8''} \ominus \frac{\partial}{\partial p_8'''} \right)}_{\alpha\beta\gamma} + \underbrace{p_3 p_5 \left( \frac{\partial}{\partial p_8'''} \ominus \frac{\partial}{\partial p_8} \ominus \frac{\partial}{\partial p_8'''} \right)}_{\alpha\gamma\delta} \\ K_4 &= p_3 p_3'' \left( \frac{\partial}{\partial p_6''} \ominus \frac{\partial}{\partial p_6} \right) + p_2 p_3'' \frac{\partial}{\partial p_5} \ominus p_2 p_3 \frac{\partial}{\partial p_5''} \end{aligned} \quad (29)$$

Next we convert these bosonic operators into the BRST one, depending on Grassmannian variables with appropriately chosen signs in front of the different items to guarantee the nilpotency. This provides the collection of differentials.

We do not write the lengthy expressions for unreduced differentials, and proceed directly to reduced ones, which in the case of 4-vertex graphs are still slightly more concise.

### 3.5 Reductions

This time not all the edges are the same, and it looks like at least two different reductions could be performed, for the marked edges  $A$  (or  $B$ ,  $F$ ,  $G$ ) and  $H$  (or  $C$ ,  $D$ ,  $E$ ).

The following table shows by crosses the time-variables which are reduced (their  $\theta$ -components eliminated, only  $\eta$ -components left) when different edges are marked. Surviving variables correspond to empty spaces in the table. The numbers of eliminated and survived variables are shown in the last two lines respectively. Clearly, the *a priori* (topologically) equivalent reductions imply nullification of very different  $\theta$ -variables (all the 20  $\eta$ 's are always present), thus the reduced differentials are different and it is not technically trivial that the resulting cohomologies are the same.

It is even less trivial that the two *a priori* different reductions, which even leave different numbers of  $p$ -variables (8 and 6) in the differentials, give rise to the same reduced superpolynomials.

	A	B	C	D	E	F	G	H
$p_2 = AB$	$x$	$x$						
$p'_2 = FG$						$x$	$x$	
$p_3 = DEF$				$x$	$x$	$x$		
$p'_3 = AEH$	$x$				$x$			$x$
$p''_3 = CGH$			$x$				$x$	$x$
$p'''_3 = BB CD$		$x$	$x$	$x$				
$p_4 = CDEH$			$x$	$x$	$x$			$x$
$p_5 = ABCGH$	$x$	$x$	$x$				$x$	$x$
$p'_5 = BCGFD$		$x$	$x$	$x$		$x$	$x$	
$p''_5 = ABDFE$	$x$	$x$		$x$	$x$	$x$		
$p'''_5 = AHGFE$	$x$				$x$	$x$	$x$	$x$
$p_6 = CDEFGH$			$x$	$x$	$x$	$x$	$x$	$x$
$p'_6 = ABDCHE$	$x$	$x$	$x$	$x$	$x$			$x$
$p''_6 = CGFDEH$			$x$	$x$	$x$	$x$	$x$	$x$
$p'''_6 = ABHEDC$	$x$	$x$	$x$	$x$	$x$			$x$
all five $p_8$	$x$	$x$	$x$	$x$	$x$	$x$	$x$	$x$
	12	12	14	14	14	12	12	14
	8	8	6	6	6	8	8	6

Before proceeding further, we should check the consistency of our reductions – that derivatives w.r.t. reduced variables are always multiplied by reduced variables and thus do not contribute. Let us show how this works for the first operator  $K_1$  in the more involved case of  $4_1$ . The factor  $(p'_6 + p'''_6)$  is reduced ( $\theta'_6$  and  $\theta'''_6$  vanish) in all the reductions, except for  $F$  and  $G$  – but in those cases neither  $p'_3$  nor  $p'''_3$  is vanishing: this makes all the reductions of the first item in  $K_1$  consistent. Likewise, in the second item  $p'_5$  is reduced in all cases except for  $A$ ,  $E$  and  $H$ , but neither  $p'_2$  nor  $p'''_3$  are reduced. The same happens in the third item: when  $p'''_5$  remains unreduced (in  $B$ ,  $C$  and  $D$  reductions), neither  $p'_2$  nor  $p'_3$  are reduced. In a similar way for both  $4_1$  and  $3_1$  one can check self-consistency of all eight reductions of all the four operators  $K$  – and of their super-counterparts – the differentials  $d$ . All this will be transparently seen in explicit calculations below – but those we do only for one of the eight: for  $A$ -reduction.

## 4 $A$ -reduction of the cut-and-join operator for the $3_1$ knot

First of all, we write the cut-and-join operators (29), underlying the reduced variables  $p$ : this means that the corresponding Grassmannian variable  $\vartheta$  contains only  $\eta$ -, but not the  $\theta$ -component.

### 4.1 Reduction of bosonic operator (29)

$$\begin{aligned}
\underline{K}_1 &= (p_6 + p''_6) \frac{\partial^2}{\partial p'_2 \partial p_4} + (\underline{p'_6} + \underline{p'''_6}) \frac{\partial^2}{\partial \underline{p_2} \partial \underline{p_4}} \\
\underline{K}_2 &= p_3 p''_3 \left( \frac{\partial}{\partial p'_6} \ominus \frac{\partial}{\partial p_6} \right) + \underline{p'_3} p'''_3 \left( \frac{\partial}{\partial \underline{p'''_6}} \ominus \frac{\partial}{\partial \underline{p'_6}} \right) + \underline{p_8} \left( \frac{\partial^2}{\partial p'_2 \partial \underline{p'''_6}} \ominus \frac{\partial^2}{\partial \underline{p_2} \partial p_6} \right) + \\
&+ \underline{p'_8} \left( \frac{\partial^2}{\partial p'_2 \partial \underline{p'_6}} \ominus \frac{\partial^2}{\partial \underline{p_2} \partial p_6} \right) + \underline{p''_8} \left( \frac{\partial^2}{\partial \underline{p_2} \partial p'_6} \ominus \frac{\partial^2}{\partial p'_2 \partial \underline{p'_6}} \right) + \underline{p'''_8} \left( \frac{\partial^2}{\partial p'_2 \partial \underline{p'''_6}} \ominus \frac{\partial^2}{\partial \underline{p_2} \partial p'_6} \right) \\
\underline{K}_3 &= p'_3 \underline{p''_5} \left( \frac{\partial}{\partial \underline{p'_8}} + \frac{\partial}{\partial \underline{p''_8}} \right) \ominus \underline{p''_5} \frac{\partial^2}{\partial \underline{p_2} \partial p_3} + p'''_3 \underline{p'''_5} \left( \frac{\partial}{\partial \underline{p'_8}} \ominus \frac{\partial}{\partial \underline{p_8}} \right) + \underline{p'''_5} \frac{\partial^2}{\partial p'_2 \partial \underline{p'_3}} + \\
&+ p_3 \underline{p_5} \left( \frac{\partial}{\partial \underline{p'''_8}} \ominus \frac{\partial}{\partial \underline{p_8}} \right) + \underline{p_5} \frac{\partial^2}{\partial \underline{p_2} \partial p'_3} + \underline{p'_3} p'_5 \left( \frac{\partial}{\partial \underline{p''_8}} + \frac{\partial}{\partial \underline{p'''_8}} \right) \ominus p'_5 \frac{\partial^2}{\partial p'_2 \partial \underline{p'''_3}}
\end{aligned}$$

$$\underline{K}_4 = \underline{p}_8''' \left( \ominus \frac{\partial^2}{\partial p_3 \partial \underline{p}_5} + \frac{\partial^2}{\partial \underline{p}_3' \partial p_5'} \ominus \frac{\partial^2}{\partial p_3' \partial \underline{p}_5''} + \frac{\partial^2}{\partial \underline{p}_3''' \partial p_5''} \right) \quad (30)$$

These formulas allow us to write the  $A$ -reduced differentials in the  $3_1$  case and calculate their cohomologies.

## 4.2 Differential $\underline{d}_1$ and the cohomology $\underline{H}_0$

From

$$\underline{K}_1 = (p_6 + p_6'') \frac{\partial^2}{\partial p_2' \partial p_4} + (\underline{p}_6' + \underline{p}_6''') \frac{\partial^2}{\partial \underline{p}_2' \partial p_4} \quad (31)$$

we obtain:

$$\underline{d}_1 = -(\theta_6 + \theta_6'') \left( \frac{\partial^2}{\partial \theta_2' \partial \eta_4} + \frac{\partial^2}{\partial \eta_2' \partial \theta_4} \right) - (\eta_6 + \eta_6'') \frac{\partial^2}{\partial \eta_2' \partial \eta_4} + (\eta_6' + \eta_6''') \frac{\partial^2}{\partial \eta_2' \partial \eta_4} \quad (32)$$

Note non-trivial signs in this formula: they are such that the  $\underline{d}_1$ -images of all basis vectors of the 4-dimensional space  $\underline{\mathcal{C}}_1 = \underline{V}_2 \otimes V_2' \otimes V_4$  (its quantum dimension is  $\dim_q(\underline{\mathcal{C}}_1) = qD^2$ ) are expressed through the basis vectors of  $\underline{\mathcal{C}}_2$  with non-negative coefficients (we remind our convention  $\frac{\partial^2}{\partial \theta \partial \eta}(\theta \eta) = +1$ ):

$$\underline{d}_1 \downarrow \begin{array}{c} \underline{\mathcal{C}}_0 \\ \underline{d}_1 \underline{\mathcal{C}}_0 \end{array} \left\| \begin{array}{cc} \eta_2 \theta_2' \theta_4 & \eta_2 \theta_2' \eta_4 \\ 0 & \eta_2(\theta_6 + \theta_6'') + \theta_2'(\eta_6' + \eta_6''') \end{array} \right\| \begin{array}{cc} \eta_2 \eta_2' \theta_4 & \eta_2 \eta_2' \eta_4 \\ \eta_2(\theta_6 + \theta_6'') & \eta_2(\eta_6 + \eta_6'') + \eta_2'(\eta_6' + \eta_6''') \end{array} \right\|$$

Clearly,

$$\text{Ker}(\underline{d}_1) = \{\eta_2 \theta_2' \theta_4\}, \quad \dim_q(\underline{H}_0) = \dim_q \text{Ker}(\underline{d}_1) = q^{-1} \quad (33)$$

The image

$$\text{Im}(\underline{d}_1) = \left\{ \eta_2(\theta_6 + \theta_6''), \theta_2'(\eta_6' + \eta_6'''), \eta_2(\eta_6 + \eta_6'') + \eta_2'(\eta_6' + \eta_6''') \right\}, \quad \dim_q \text{Im}(\underline{d}_1) = 2 + q^2 \quad (34)$$

Note that differential itself is not really needed in this calculation: what we need are the vector spaces with bases and bosonic cut-and-join operator with the  $\ominus$  labels – this is enough to construct the kernel and image of the differential, without knowing its explicit form (the sign assignments).

## 4.3 Differential $\underline{d}_2$ and the cohomology $\underline{H}_1$

This time we begin with

$$\begin{aligned} \underline{K}_2 = & p_3 p_3'' \left( \frac{\partial}{\partial p_6''} \ominus \frac{\partial}{\partial p_6} \right) + \underline{p}_3' p_3''' \left( \frac{\partial}{\partial \underline{p}_6'''} \ominus \frac{\partial}{\partial \underline{p}_6'} \right) + \underline{p}_8 \left( \frac{\partial^2}{\partial p_2' \partial \underline{p}_6'''} \ominus \frac{\partial^2}{\partial \underline{p}_2' \partial p_6''} \right) + \\ & + \underline{p}_8' \left( \frac{\partial^2}{\partial p_2' \partial \underline{p}_6'} \ominus \frac{\partial^2}{\partial \underline{p}_2' \partial p_6} \right) + \underline{p}_8'' \left( \frac{\partial^2}{\partial \underline{p}_2' \partial p_6''} \ominus \frac{\partial^2}{\partial p_2' \partial \underline{p}_6''} \right) + \underline{p}_8''' \left( \frac{\partial^2}{\partial p_2' \partial \underline{p}_6'''} \ominus \frac{\partial^2}{\partial \underline{p}_2' \partial p_6'''} \right) \end{aligned} \quad (35)$$

The signs in the differential  $\underline{d}_2$  are now adjusted in two steps: first, we choose the signs so that basis vectors are mapped into basis vectors with non-negative coefficients, second we reverse the signs in the terms, marked by the  $\ominus$  signs. As usual these terms will be marked by a hat. In this way we get:

$$\begin{aligned} \underline{d}_2 = & -\theta_3 \theta_3'' \left( \frac{\partial}{\partial \theta_6''} \hat{-} \frac{\partial}{\partial \theta_6} \right) - (\theta_3 \eta_3'' + \eta_3 \theta_3'') \left( \frac{\partial}{\partial \eta_6''} \hat{-} \frac{\partial}{\partial \eta_6} \right) - \eta_3' \theta_3''' \left( \frac{\partial}{\partial \eta_6'''} \hat{-} \frac{\partial}{\partial \eta_6'} \right) + \\ & + \eta_8 \left( \frac{\partial^2}{\partial \eta_2' \partial \eta_6'''} \hat{-} \frac{\partial^2}{\partial \eta_2' \partial \eta_6} \right) + \eta_8' \left( \frac{\partial^2}{\partial \eta_2' \partial \eta_6'} \hat{-} \frac{\partial^2}{\partial \eta_2' \partial \eta_6''} \right) + \\ & + \eta_8'' \left( \frac{\partial^2}{\partial \eta_2' \partial \eta_6''} \hat{-} \frac{\partial^2}{\partial \eta_2' \partial \eta_6'''} \right) + \eta_8''' \left( \frac{\partial^2}{\partial \eta_2' \partial \eta_6'''} \hat{-} \frac{\partial^2}{\partial \eta_2' \partial \eta_6''} \right) \end{aligned} \quad (36)$$



## 4.5 Differential $\underline{d}_4$ and the cohomologies $\underline{H}_3$ and $\underline{H}_4$

Finally, from

$$\underline{K}_4 = \underline{p}_8''' \left( \ominus \frac{\partial^2}{\partial p_3 \partial p_5} + \frac{\partial^2}{\partial p_3' \partial p_5'} \ominus \frac{\partial^2}{\partial p_3'' \partial p_5''} + \frac{\partial^2}{\partial p_3''' \partial p_5'''} \right) \quad (43)$$

$$\underline{d}_4 = \eta_8''' \left( \hat{-} \frac{\partial^2}{\partial \eta_3 \partial \eta_5} + \frac{\partial^2}{\partial \eta_3' \partial \eta_5'} \hat{-} \frac{\partial^2}{\partial \eta_3'' \partial \eta_5''} + \frac{\partial^2}{\partial \eta_3''' \partial \eta_5'''} \right) \quad (44)$$

Thus in the  $4 \cdot 2 = 8$ -dimensional space  $C_4 = V_3 \otimes \underline{V}_5 \oplus \underline{V}_3 \otimes V_5 \oplus V_3 \otimes \underline{V}_5 \oplus \underline{V}_3 \otimes V_5$  the 7-dimensional kernel

$$\ker(\underline{d}_4) = \left\{ \theta_3 \eta_5, \eta_3' \theta_5', \theta_3'' \eta_5'', \theta_3''' \eta_5''', \eta_3 \eta_5 + \eta_3' \eta_5', \eta_3 \eta_5 - \eta_3'' \eta_5'', \eta_3' \eta_5' - \eta_3''' \eta_5''' \right\} \quad (45)$$

Therefore

$$\underline{H}_3 = \text{Ker}(\underline{d}_4) / \text{Im}(\underline{d}_3) = \left\{ \eta_3 \eta_5 + \eta_3' \eta_5' \right\} \quad \text{and} \quad \dim_q(\underline{H}_3) = q^2 \quad (46)$$

The space  $C_5 = \underline{V}$  is one-dimensional, the basis element is  $\eta_8'''$  and it coincides with the  $8 - 7 = 1$ -dimensional image of  $\underline{d}_4$ . Therefore the coimage of  $\underline{d}_4$  is empty, and the cohomology

$$\underline{H}_4 = 0 \quad (47)$$

## 4.6 Reduced Jones superpolynomials for $3_1$

Thus

$$\begin{aligned} \dim_q(\underline{H}_0) &= q^{-1}, \\ \dim_q(\underline{H}_1) &= 0, \\ \dim_q(\underline{H}_2) &= q, \\ \dim_q(\underline{H}_3) &= q^2, \\ \dim_q(\underline{H}_4) &= 0 \end{aligned} \quad (48)$$

and therefore

$$\underline{P}_{\square}^{3_1} = q^{-1} \cdot q^4 \sum_{i=0}^4 (qT)^i \cdot \dim_q(\underline{H}_i) = \frac{q^4}{q} \left( q^{-1} + 0 \cdot (qT) + q^1 \cdot (qT)^2 + q^2 \cdot (qT)^3 \right) = \boxed{q^2 + q^6 T^2 + q^8 T^3} \quad (49)$$

what coincides with the answer (21), obtained from the 2-strand representation.

## 5 A-reduced differentials for the $4_1$ knot

Now we repeat the same procedure in the case of the figure-eight knot.

### 5.1 Reduction of bosonic operator (30)

$$\underline{K}_1 = (\underline{p}_6''' \ominus \underline{p}_6') \frac{\partial^2}{\partial p_3' \partial p_3'''} \ominus p_5' \frac{\partial^2}{\partial p_2' \partial p_3'''} + \underline{p}_5''' \frac{\partial}{\partial p_2' \partial p_3'}$$

$$\begin{aligned} \underline{K}_2 &= \underline{p}_2 p_4 \frac{\partial}{\partial p_6'} + \underline{p}_2 p_4 \frac{\partial}{\partial p_6'''} + \underline{p}_8 \left( \frac{\partial^2}{\partial p_2' \partial p_6'''} \ominus \frac{\partial^2}{\partial p_3''' \partial p_5'''} \right) + \underline{p}_8' \left( \frac{\partial^2}{\partial p_2' \partial p_6'} + \frac{\partial^2}{\partial p_3''' \partial p_5'''} \right) + \\ &+ \underline{p}_8'' \left( \frac{\partial^2}{\partial p_3' \partial p_5'} \ominus \frac{\partial^2}{\partial p_2' \partial p_6'} \right) + \underline{p}_8''' \left( \frac{\partial^2}{\partial p_2' \partial p_6'''} + \frac{\partial^2}{\partial p_3' \partial p_5'} \right) + \underline{p}_8'''' \left( \frac{\partial^2}{\partial p_3' \partial p_5'} + \frac{\partial^2}{\partial p_3''' \partial p_5'''} \right) \end{aligned}$$

$$\begin{aligned}
\underline{K}_3 = & p_6 \frac{\partial^2}{\partial p_2' \partial p_4} \ominus \underline{p}_2 p_6 \left( \frac{\partial}{\partial p_8} + \frac{\partial}{\partial p_8'} \right) + p_6'' \frac{\partial^2}{\partial p_2' \partial p_4} + \underline{p}_2 p_6'' \left( \frac{\partial}{\partial p_8''} \ominus \frac{\partial}{\partial p_8'''} \right) + \\
& + p_3'' \underline{p}_5'' \left( \frac{\partial}{\partial p_8'} + \frac{\partial}{\partial p_8''} \ominus \frac{\partial}{\partial p_8'''} \right) + p_3 \underline{p}_5 \left( \frac{\partial}{\partial p_8'''} \ominus \frac{\partial}{\partial p_8} \ominus \frac{\partial}{\partial p_8'''} \right) \\
\underline{K}_4 = & p_3 p_3'' \left( \frac{\partial}{\partial p_6''} \ominus \frac{\partial}{\partial p_6} \right) + \underline{p}_2 p_3'' \frac{\partial}{\partial p_5} \ominus \underline{p}_2 p_3 \frac{\partial}{\partial p_5''}
\end{aligned} \tag{50}$$

## 5.2 Differential $\underline{d}_1$ and the cohomology $\underline{H}_0$

As usual, we begin from the relevant bosonic operator

$$\underline{K}_1 = (\underline{p}_6''' \ominus \underline{p}_6') \frac{\partial^2}{\partial p_3' \partial p_3'''} \ominus p_5' \frac{\partial^2}{\partial p_2' \partial p_3'''} + \underline{p}_5''' \frac{\partial}{\partial p_2' \partial p_3'} \tag{51}$$

In this case the space  $\underline{\mathcal{C}}_0 = V_2' \otimes \underline{V}_3' \otimes V_3'''$  has dimension  $\dim_q(\underline{\mathcal{C}}_0) = q^{-1} + 2q + q^3 = qD^2$ . The differential

$$\underline{d}_1 = -(\eta_6''' - \eta_6') \frac{\partial^2}{\partial \eta_3' \partial \eta_3'''} \hat{\wedge} \theta_5' \left( \frac{\partial^2}{\partial \theta_2' \partial \eta_3'''} + \frac{\partial^2}{\partial \eta_2' \partial \theta_3'''} \right) \hat{\wedge} \eta_5' \frac{\partial^2}{\partial \eta_2' \partial \eta_3'''} - \eta_5''' \frac{\partial^2}{\partial \eta_2' \partial \eta_3'} \tag{52}$$

converts the basis in this space as follows:

$$\underline{d}_1 \downarrow \begin{array}{c} \underline{\mathcal{C}}_0 \\ \underline{d}_1 \underline{\mathcal{C}}_0 \end{array} \left\| \begin{array}{c} \theta_2' \eta_3' \theta_3''' \\ 0 \end{array} \right| \begin{array}{c} \theta_2' \eta_3' \eta_3''' \\ \theta_2' (\eta_6''' \hat{\wedge} \eta_6') \hat{\wedge} \eta_3' \theta_5' \end{array} \left| \begin{array}{c} \eta_2' \eta_3' \theta_3''' \\ \theta_3''' \eta_5''' \hat{\wedge} \theta_3' \eta_5' \end{array} \right| \begin{array}{c} \eta_2' \eta_3' \eta_3''' \\ \eta_2' (\eta_6''' \hat{\wedge} \eta_6') \hat{\wedge} \eta_3' \eta_5' + \eta_3''' \eta_5''' \end{array} \tag{53}$$

Clearly,

$$\text{Ker}(\underline{d}_1) = \{\theta_2' \eta_3' \theta_3'''\}, \quad \dim_q(\underline{H}_0) = \dim_q \text{Ker}(\underline{d}_1) = q^{-1} \tag{54}$$

Note that the second item in  $\underline{d}_1$  would annihilate also the linear combination  $\theta_2' \eta_3' \eta_3''' - \eta_2' \eta_3' \theta_3'''$ , but it is not annihilated by the first item in  $\underline{d}_1$ . The image

$$\text{Im}(\underline{d}_1) = \left\{ \theta_2' (\eta_6''' \hat{\wedge} \eta_6''') + \eta_3' \theta_5', \quad \theta_3' \eta_5' \hat{\wedge} \theta_3''' \eta_5''', \quad \eta_2' (\eta_6''' \hat{\wedge} \eta_6''') + \eta_3' \eta_5' \hat{\wedge} \eta_3''' \eta_5''' \right\}, \quad \dim_q \text{Im}(\underline{d}_1) = 2 + q^2 \tag{55}$$

## 5.3 Differential $\underline{d}_2$ and the cohomology $\underline{H}_1$

From

$$\begin{aligned}
\underline{K}_2 = & \underline{p}_2 p_4 \frac{\partial}{\partial p_6'} + \underline{p}_2 p_4 \frac{\partial}{\partial p_6'''} + \underline{p}_8 \left( \frac{\partial^2}{\partial p_2' \partial p_6'''} \ominus \frac{\partial^2}{\partial p_3''' \partial p_5'''} \right) + \underline{p}_8' \left( \frac{\partial^2}{\partial p_2' \partial p_6'} + \frac{\partial^2}{\partial p_3''' \partial p_5'''} \right) + \\
& + \underline{p}_8'' \left( \frac{\partial^2}{\partial p_3' \partial p_5'} \ominus \frac{\partial^2}{\partial p_2' \partial p_6'} \right) + \underline{p}_8''' \left( \frac{\partial^2}{\partial p_2' \partial p_6'''} + \frac{\partial^2}{\partial p_3' \partial p_5'} \right) + \underline{p}_8'''' \left( \frac{\partial^2}{\partial p_3' \partial p_5'} + \frac{\partial^2}{\partial p_3''' \partial p_5'''} \right)
\end{aligned} \tag{56}$$

we get the differential:

$$\begin{aligned}
\underline{d}_2 = & \eta_2 \theta_4 \left( \frac{\partial}{\partial \eta_6'} + \frac{\partial}{\partial \eta_6'''} \right) + \eta_8 \left( \frac{\partial^2}{\partial \eta_2' \partial \eta_6'''} \hat{\wedge} \frac{\partial^2}{\partial \eta_3''' \partial \eta_5'''} \right) + \eta_8' \left( \frac{\partial^2}{\partial \eta_2' \partial \eta_6'} + \frac{\partial^2}{\partial \eta_3''' \partial \eta_5'''} \right) + \\
& + \eta_8'' \left( \frac{\partial^2}{\partial \eta_3' \partial \eta_5'} \hat{\wedge} \frac{\partial^2}{\partial \eta_2' \partial \eta_6'} \right) + \eta_8''' \left( \frac{\partial^2}{\partial \eta_2' \partial \eta_6'''} + \frac{\partial^2}{\partial \eta_3' \partial \eta_5'} \right) + \eta_8'''' \left( \frac{\partial^2}{\partial \eta_3' \partial \eta_5'} + \frac{\partial^2}{\partial \eta_3''' \partial \eta_5'''} \right)
\end{aligned} \tag{57}$$

which acts on the space  $\underline{\mathcal{C}}_1$  of dimension  $\dim_q(\underline{\mathcal{C}}_1) = 4 + 4q^2 = 4qD$  as follows:

$$\underline{d}_2 \downarrow \begin{array}{c} \underline{\mathcal{C}}_1 \\ \underline{d}_2 \underline{\mathcal{C}}_1 \end{array} \left\| \begin{array}{c} \theta_2' \eta_6' \\ \eta_2 \theta_2' \eta_4 \end{array} \right| \begin{array}{c} \eta_2' \eta_6' \\ \eta_2 \eta_2' \theta_4 + \eta_8' \hat{\wedge} \eta_8'' \end{array} \left| \begin{array}{c} \theta_2' \eta_6''' \\ \eta_2 \theta_2' \eta_4 \end{array} \right| \begin{array}{c} \eta_2' \eta_6''' \\ \eta_2 \eta_2' \theta_4 + \eta_8 + \eta_8''' \end{array} \left| \begin{array}{c} \eta_3' \theta_5' \\ 0 \end{array} \right| \begin{array}{c} \eta_3' \eta_5' \\ \eta_8'' + \eta_8''' + \eta_8'''' \end{array} \left| \begin{array}{c} \theta_3''' \eta_5''' \\ 0 \end{array} \right| \begin{array}{c} \eta_3''' \eta_5''' \\ \hat{\wedge} \eta_8 + \eta_8' + \eta_8''' \end{array} \tag{58}$$

so that

$$\begin{aligned} \text{Ker}(\underline{d}_2) &= \left\{ \eta'_3 \theta'_5, \quad \theta'''_3 \eta'''_5, \quad \theta'_2(\eta'_6 - \eta'''_6), \quad \eta'_2(\eta'_6 - \eta'''_6) + \eta'_3 \eta'_5 - \eta'''_3 \eta'''_5 \right\}, \\ \dim_q \text{Ker}(\underline{d}_2) &= 3 + q^2, \quad \underline{H}_1 = \text{Ker}(\underline{d}_2)/\text{Im}(\underline{d}_1) = \left\{ \theta'_3 \eta'_5 \right\}, \quad \dim_q(\underline{H}_1) = 1 \end{aligned} \quad (58)$$

and

$$\begin{aligned} \text{Im}(\underline{d}_2) &= \left\{ \eta_2 \theta'_2 \theta_4, \quad \eta_2 \eta'_2 \theta_4 + \eta_8 + \eta'''_8, \quad \eta_8 \hat{\eta}'_8 \hat{\eta}'''_8, \quad \eta''_8 + \eta'''_8 + \eta''''_8 \right\}, \\ \dim_q \text{Im}(\underline{d}_2) &= q^{-1} + 3q, \quad \dim_q \text{Ker}(\underline{d}_2) + q \cdot \dim_q \text{Im}(\underline{d}_2) = \dim_q(\mathcal{C}_1) = 4 + 4q^2 = 4qD \end{aligned} \quad (59)$$

The combination  $\eta_8 - \eta'_8 \hat{\eta}''_8 + \eta'''_8 \in \text{Im}(\underline{d}_2)$  as a linear combination of the two last entries in (59).

#### 5.4 Differential $\underline{d}_3$ and the cohomology $\underline{H}_2$

From

$$\begin{aligned} \underline{K}_3 &= p_6 \frac{\partial^2}{\partial p'_2 \partial p_4} \ominus \underline{p}_2 p_6 \left( \frac{\partial}{\partial \underline{p}_8} + \frac{\partial}{\partial \underline{p}'_8} \right) + p''_6 \frac{\partial^2}{\partial p'_2 \partial p_4} + \underline{p}_2 p''_6 \left( \frac{\partial}{\partial \underline{p}''_8} \ominus \frac{\partial}{\partial \underline{p}'''_8} \right) + \\ &+ p'''_3 \underline{p}_5 \left( \frac{\partial}{\partial \underline{p}'_8} + \frac{\partial}{\partial \underline{p}''_8} \ominus \frac{\partial}{\partial \underline{p}'''_8} \right) + p_3 \underline{p}_5 \left( \frac{\partial}{\partial \underline{p}'''_8} \ominus \frac{\partial}{\partial \underline{p}_8} \ominus \frac{\partial}{\partial \underline{p}''''_8} \right) \end{aligned} \quad (60)$$

we get:

$$\begin{aligned} \underline{d}_3 &= -(\theta_6 + \theta''_6) \left( \frac{\partial^2}{\partial \theta'_2 \partial \eta_4} + \frac{\partial^2}{\partial \eta'_2 \partial \theta_4} \right) - (\eta_6 + \eta''_6) \frac{\partial^2}{\partial \eta'_2 \partial \eta_4} + \\ &\hat{\eta}_2 \theta_6 \left( \frac{\partial}{\partial \eta_8} + \frac{\partial}{\partial \eta'_8} \right) + \eta_2 \theta''_6 \left( \frac{\partial}{\partial \eta''_8} \hat{\eta} \frac{\partial}{\partial \eta'''_8} \right) + \eta_3 \theta_5 \left( \frac{\partial}{\partial \eta'''_8} \hat{\eta} \frac{\partial}{\partial \eta_8} \hat{\eta} \frac{\partial}{\partial \eta'''_8} \right) + \eta'''_3 \theta''_5 \left( \frac{\partial}{\partial \eta''_8} + \frac{\partial}{\partial \eta'_8} \hat{\eta} \frac{\partial}{\partial \eta'''_8} \right) \end{aligned} \quad (61)$$

acts on the space  $\underline{\mathcal{C}}_2$  of dimension  $\dim_q(\underline{\mathcal{C}}_2) = q^{-1} + (2+5)q + q^3 = q(D^2 + 5)$ :

$$\begin{array}{c} \underline{d}_3 \downarrow \\ \underline{d}_3 \underline{\mathcal{C}}_2 \parallel \left\| \begin{array}{c|c|c|c|c} \eta_2 \theta'_2 \theta_4 & \eta_2 \theta'_2 \eta_4 & \eta_2 \eta'_2 \theta_4 & \eta_2 \eta'_2 \eta_4 & \\ \hline 0 & \eta_2(\theta_6 + \theta''_6) & \eta_2(\theta_6 + \theta''_6) & \eta_2(\eta_6 + \eta''_6) & \end{array} \right\| \\ \\ \left\| \begin{array}{c|c|c|c|c} \eta_8 & \eta'_8 & \eta''_8 & \eta'''_8 & \eta''''_8 \\ \hline \hat{\eta}_2 \theta_6 \hat{\eta}_3 \theta_5 & \hat{\eta}_2 \theta_6 + \eta'''_3 \theta''_5 & \eta_2 \theta''_6 + \eta'''_3 \theta''_5 & \hat{\eta}_2 \theta''_6 + \eta_3 \theta_5 & \hat{\eta}_3 \theta_5 \hat{\eta}_3 \theta''_5 \end{array} \right\| \end{array}$$

Therefore

$$\begin{aligned} \text{Ker}(\underline{d}_3) &= \left\{ \eta_2 \theta'_2 \theta_4, \quad \eta_2(\theta'_2 \eta_4 - \eta'_2 \theta_4), \quad \eta_2 \eta'_2 \theta_4 + \eta_8 + \eta'''_8, \quad \eta_8 - \eta'_8 - \eta'''_8, \quad \eta''_8 + \eta'''_8 + \eta''''_8 \right\}, \\ \dim_q \text{Ker}(\underline{d}_3) &= q^{-1} + 4q, \\ \underline{H}_2 &= \text{Ker}(\underline{d}_3)/\text{Im}(\underline{d}_2) = \left\{ \eta_2(\theta'_2 \eta_4 - \eta'_2 \theta_4) \right\}, \quad \dim_q(\underline{H}_2) = q \end{aligned} \quad (62)$$

and

$$\begin{aligned} \text{Im}(\underline{d}_3) &= \left\{ \eta_2(\theta_6 + \theta''_6), \quad \eta_3 \theta_5 + \eta'''_3 \theta''_5, \quad \eta_2 \theta_6 + \eta_3 \theta_5, \quad \eta_2(\eta_6 + \eta''_6) \right\}, \\ \dim_q \text{Im}(\underline{d}_3) &= 3 + q^2, \quad \dim_q \text{Ker}(\underline{d}_3) + q \cdot \dim_q \text{Im}(\underline{d}_3) = \dim_q(\mathcal{C}_2) = q^{-1} + 7q + q^3 \end{aligned} \quad (63)$$

#### 5.5 Differential $\underline{d}_4$ and the cohomologies $\underline{H}_3$ and $\underline{H}_4$

Finally,

$$\underline{K}_4 = p_3 p''_3 \left( \frac{\partial}{\partial p''_6} \ominus \frac{\partial}{\partial p_6} \right) + \underline{p}_2 p''_3 \frac{\partial}{\partial \underline{p}_5} \ominus \underline{p}_2 p_3 \frac{\partial}{\partial \underline{p}''_5} \quad (64)$$

The corresponding differential

$$\underline{d}_4 = -\theta_3\theta_3'' \left( \frac{\partial}{\partial\theta_6''} - \frac{\partial}{\partial\theta_6} \right) - (\theta_3\eta_3'' - \eta_3\theta_3'') \left( \frac{\partial}{\partial\eta_6''} - \frac{\partial}{\partial\eta_6} \right) + \eta_2\theta_3'' \frac{\partial}{\partial\eta_5} + \eta_2\theta_3 \frac{\partial}{\partial\eta_5''} \quad (65)$$

converts the  $\underline{\mathcal{C}}_3$  of dimension  $\dim_q(\underline{\mathcal{C}}_3) = 4 + 4q^2 = 4qD$ :

$$\underline{d}_4 \downarrow \quad \begin{array}{c} \underline{\mathcal{C}}_3 \\ \hline \underline{d}_4 \underline{\mathcal{C}}_3 \end{array} \left\| \begin{array}{c|c|c|c|c|c|c|c|c|c} \eta_2\theta_6 & \eta_2\eta_6 & \eta_2\theta_6'' & \eta_2\eta_6'' & \theta_3\eta_5 & \eta_3\eta_5 & \theta_3''\eta_5'' & \eta_3''\eta_5'' \\ \hline \hat{-} \eta_2\theta_3\theta_3'' & \hat{-} \eta_2(\theta_3\eta_3'' + \eta_3\theta_3'') & \eta_2\theta_3\theta_3'' & \eta_2(\theta_3\eta_3'' + \eta_3\theta_3'') & \eta_2\theta_3\theta_3'' & \eta_2\eta_3\theta_3'' & \hat{-} \eta_2\theta_3\theta_3'' & \hat{-} \eta_2\theta_3\eta_3'' \end{array} \right.$$

Thus

$$\begin{aligned} \text{Ker}(\underline{d}_4) &= \left\{ \eta_2(\theta_6 + \theta_6''), \quad \theta_3\eta_5 + \theta_3''\eta_5'', \quad \eta_2\theta_6 + \theta_3\eta_5, \quad \eta_2\eta_6 + \eta_3\eta_5 - \eta_3''\eta_5'', \quad \eta_2(\eta_6 + \eta_6'') \right\}, \\ \dim_q \text{Ker}(\underline{d}_4) &= 3 + 2q^2, \\ \underline{H}_3 &= \text{Ker}(\underline{d}_4)/\text{Im}(\underline{d}_3) = \left\{ \eta_2\eta_6 + \eta_3\eta_5 - \eta_3''\eta_5'' \right\}, \quad \dim_q(\underline{H}_3) = q^2 \end{aligned} \quad (66)$$

The space

$$\mathcal{C}_4 = \left\{ \eta_2\theta_3\theta_3'', \quad \eta_2\theta_3\eta_3'', \quad \eta_2\eta_3\theta_3'', \quad \eta_2\eta_3\eta_3'' \right\}, \quad \dim_q(\underline{\mathcal{C}}_4) = q^{-1} + 2q + q^3 = qD^2 \quad (67)$$

$$\begin{aligned} \text{Im}(\underline{d}_4) &= \left\{ \eta_2\theta_3\theta_3'', \quad \eta_2\theta_3\eta_3'', \quad \eta_2\eta_3\theta_3'' \right\} \\ \dim_q \text{Im}(\underline{d}_4) &= q^{-1} + 2q, \\ \text{Coim}(\underline{d}_4) &= \{ \eta_2\eta_3\eta_3'' \}, \quad \dim_q(\underline{H}_4) = \dim_q \text{Coim}(\underline{d}_4) = q^3 \end{aligned} \quad (68)$$

## 5.6 Reduced Jones superpolynomials for $4_1$

Thus we have

$$\begin{aligned} \dim_q(\underline{H}_0) &= q^{-1}, \\ \dim_q(\underline{H}_1) &= 1, \\ \dim_q(\underline{H}_2) &= q, \\ \dim_q(\underline{H}_3) &= q^2, \\ \dim_q(\underline{H}_4) &= q^3 \end{aligned} \quad (69)$$

so that the reduced Jones superpolynomial is

$$\begin{aligned} P_{\square}^{4_1} &= q^{-1} \frac{q^2}{(q^2T)^2} \sum_{i=0}^4 (qT)^i \cdot \dim_q(\underline{H}_i) = \frac{1}{q} \frac{q^2}{(q^2T)^2} \left( q^{-1} + (qT) + q(qT)^2 + q^2(qT)^3 + q^3(qT)^4 \right) = \\ &= \frac{1}{q^4T^2} + \frac{1}{q^2T} + 1 + q^2T + q^4T^2 \end{aligned} \quad (70)$$

what coincides with the value of the superpolynomial [22, 34]

$$\boxed{P_{\square}^{4_1}(a|q|T) = 1 + T^2a^2 + q^2T + \frac{1}{q^2T} + \frac{1}{T^2a^2}} \quad (71)$$

at  $a = q^2$ .

## 6 Conclusion

This second part of review series explains the difference between reduced and unreduced superpolynomials from the perspective of Khovanov-Rozansky categorification approach. We did not discuss the Chern-Simons-theory origin of this reduction, as well as its (in)dependence on the choice of the "marked" edge in the link diagram. Instead, since for small knots reduced polynomials are considerably simpler, we used this chance to describe cohomology calculus in much more detail than it was done in [1].



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